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Complete space-like hypersurfaces in locally symmetric Lorentz spaces

Jin Ok Baek [a], Qing-Ming Cheng [b], Young Jin Suh [a,*]

[a] *Department of Mathematics, Kyungpook National University, Taegu 702-701, South Korea*
[b] *Faculty of Science and Engineering, Department of Mathematics, Saga University, Saga 840-8502, Japan*

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Abstract

The purpose of this paper is to study complete space-like hypersurfaces with constant mean curvature in a locally symmetric Lorentz space satisfying some curvature conditions. We give an optimal estimate of the squared norm of the second fundamental form of such hypersurfaces. Furthermore, the totally umbilical hypersurfaces are characterized.

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Keywords: Constant mean curvature; Space-like hypersurface; Second fundamental form; Locally symmetric Lorentz space

1. Introduction

Let M_s^m be an m-dimensional connected semi-Riemannian manifold of index s (≥ 0). It is called a semi-definite space of index s. In particular, M_1^m is called a Lorentz space. A hypersurface M of a Lorentz space is said to be space-like if the induced metric on M from that of the Lorentz space is positive definite. When the Lorentz space M_1^m is of constant curvature c, we call it Lorentz space form, denoted by $M_1^m(c)$.

It is well known that a maximal space-like entire graph in Minkowski space $\mathbf{R}_1^{n+1}$ is a linear hyperplane [2,7]. As a generalization of the result above, Chouque-Bruhat et al.

* Corresponding author.
E-mail addresses: cheng@ms.saga-u.ac.jp (Q.-M. Cheng), yjsuh@wmail.knu.ac.kr (Y. Jin Suh).

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[10] and Ishihara [11] proved that totally geodesic hypersurfaces are the only complete space-like maximal hypersurfaces in $M_1^{n+1}(c)$, $c \geq 0$.

On the other hand, it was pointed out by Marsdan and Tipler [13] and Stumbles [19] that space-like hypersurfaces with constant mean curvature in arbitrary spacetime get interested in the relativity theory. Space-like hypersurfaces with constant mean curvature are convenient as initial hypersurfaces for the Cauchy problem in arbitrary spacetime and for studying the propagation of gravitational radiation. Hence, complete space-like hypersurfaces with constant mean curvature in a Lorentz space form $M_1^{n+1}(c)$ are extensively investigated by many differential geometers in various view points; for example, Akutagawa [1] and Ramanathan [18] proved complete space-like hypersurfaces with constant mean curvature H in the de Sitter space $S_1^{n+1}(c)$ must be totally umbilical if $n^2H^2 < 4(n-1)c$ when $n > 2$, and $H^2 \leq 1$ when $n = 2$. For further development in these directions, see [3–6,12,14,15] and so on.

As standard models of complete space-like hypersurfaces with constant mean curvature in $M_1^{n+1}(c)$, there are four classes of complete hypersurfaces $\mathbf{H}^k(c_1) \times S^{n-k}(c_2)$, $\mathbf{R}^k \times S^{n-k}(c_2)$, $\mathbf{H}^k(c_1) \times \mathbf{R}^{n-k}$ and $\mathbf{H}^k(c_1) \times \mathbf{H}^{n-k}(c_2)$, where $k = 0, 1, \ldots, n$, according to $c > 0$, $=0$ or <0. In particular, $\mathbf{H}^1(c_1) \times S^{n-1}(c_2)$ is called a hyperbolic cylinder and $\mathbf{H}^{n-1}(c_1) \times S^1(c_2)$ is called a spherical cylinder.

It is important and natural to study complete space-like hypersurfaces with constant mean curvature in the more general Lorentz spaces since they have important meaning in the relativity theory. First of all, we shall consider several examples of Lorentz spaces which are not Lorentz space forms.

Example 1. We consider the semi-Riemannian product manifold

$$H_1^k\left(-\frac{c_1}{n}\right) \times M^{n+1-k}(c_2), c_1 > 0.$$

Its sectional curvature is given by

$$K'(u_1, u_b) = -\frac{c_1}{n}, \qquad K'(u_a, u_b) = -\frac{c_1}{n}, \qquad K'(u_a, u_r) = 0, \qquad K'(u_r, u_s) = c_2,$$

where $a, b, \ldots = 2, \ldots k$, $r, s, \ldots = k+1, \ldots, n+1$, and u_1 and u_a, u_r denotes time-like and space-like vectors respectively.

Example 2. We consider the semi-Riemannian product manifold

$$R_1^k \times S^{n+1-k}(1).$$

Its sectional curvature is given by

$$K'(u_1, u_a) = 0, \qquad K'(u_a, u_b) = 0, \qquad K'(u_1, u_r) = 0, \qquad K'(u_r, u_s) = 1,$$

where $a, b, \ldots = 2, \ldots, k$ and $r, s, \ldots = k+1, \ldots, n+1$. In particular, $R_1^1 \times S^n(1)$ is so-called *Einstein Static Universe*. Of course, it is not Lorentzian space form.

Next, we shall show a general example of Lorentz space which is called *Robertson–Walker spacetime*.

Example 3. Let I denote an open interval of R_1^1 and $f > 0$ a smooth function defined on the interval I. For a three-dimensional Riemannian manifold of constant curvature c, $c = -1, 0, 1$, we construct a Lorentz space $M(c, f)$ as the warped product

$$M(c, f) = I \times_f M^3(c),$$

which is called *Robertson–Walker spacetime* (see [17], pp. 343–345). Thus, by a direct computation, we have that its sectional curvature is given by

$$K'(v, w) = \left(\frac{f'}{f}\right)^2 + \frac{c}{f^2}$$

for any space-like vectors v and w and

$$K'(e_0, v) = \frac{f''}{f}$$

for any time-like vector e_0 and any unit space-like vector v.

In this paper, we shall consider $n+1$-dimensional Lorentz spaces M' of index 1. Let ∇', K' and R' denote the semi-Riemannian connection, sectional curvature and the curvature tensor on M', respectively. For constants c_1, c_2 and c_3, we consider Lorentz spaces which satisfy the following:

(1) for any space-like vector u and any time-like vector v

$$K'(u, v) = -\frac{c_1}{n},$$

(2) for any space-like vectors u and v

$$K'(u, v) \geq c_2,$$

(3) $$|\nabla' R'| \leq \frac{c_3}{n}.$$

When M' satisfies conditions (1) and (2), we shall say that M' satisfies condition $(*)$. When M' satisfies conditions (1)–(3), we shall say that M' satisfies condition $(**)$.

Remark 1. It can be easily seen that if the Lorentz space M' is locally symmetric, then the condition (3) holds.

Remark 2. The Lorentz space form $M_1^{n+1}(c)$ satisfies the conditions $(*)$ and $(**)$, where $-(c_1/n) = c_2 = c$.

Remark 3. The Examples 1 and 2 satisfy the conditions $(*)$ and $(**)$ and the Example 3 also satisfies the conditions $(*)$ and $(**)$ if we choose an appropriate function f (see [17], pp. 343–345).

In [9,20], authors investigated complete space-like hypersurfaces M in a Lorentz space satisfying condition $(**)$. They estimated the squared norm of the second fundamental form of M under some conditions. In this paper, we shall prove the following theorem.

Theorem 1. *Let M be a complete space-like hypersurface with constant mean curvature H in an $n+1$-dimensional locally symmetric Lorentz space M' satisfying the condition* $(*)$:

(1) *If $n^2H^2 < 4(n-1)c$, where $c = 2c_2 + (c_1/n)$, then $c > 0$, $S \equiv nH^2$ and M is totally umbilical, where S denotes the squared norm of the second fundamental form of M.*

(2) *If $n^2H^2 = 4(n-1)c$, then $c \geq 0$ and either $S \equiv nH^2$ and M is totally umbilical, or* $\sup S = nc$.

(3) *If $n^2H^2 > 4(n-1)c$ and $c < 0$, then either $S \equiv nH^2$ and M is totally umbilical, or*

$$nH^2 < \sup S \leq S_{\max},$$

where $S_{\max} = n/2(n-1)[n^2H^2 - 2(n-1)c + (n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]$.

(4) *If $n^2H^2 > 4(n-1)c$ and $c \geq 0$, then either $S \equiv nH^2$ and M is totally umbilical, or*

$$S_{\max} \geq \sup S \begin{cases} > nH^2, & \text{if } H^2 \geq c, \\ \geq S_{\min}, & \text{if } H^2 < c. \end{cases}$$

where $S_{\min} = n/2(n-1)[n^2H^2 - 2(n-1)c - (n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]$.

(5) $$S \equiv \frac{n}{2(n-1)}[n^2H^2 - 2(n-1)c + (n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}],$$

if and only if M is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Theorem 2. *Let M be an n ($n > 2$) dimensional complete space-like hypersurface with constant mean curvature H in an $n+1$-dimensional locally symmetric Lorentz space M' satisfying the condition $(*)$. If the sectional curvature of M is not less than $-(c_2 + (c_1/n))$, then, $c \geq 0$, where $c = 2c_2 + (c_1/n)$. Furthermore, if $H^2 \geq c$ and*

$$S < \frac{n}{2(n-1)}[n^2H^2 - 2(n-1)c + (n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}],$$

hold, then M is totally umbilical.

Remark 4. We should notice that when M' is a Lorentz space form $M_1^{n+1}(c)$, a part of the similar results in Theorems 1 and 2 was obtained by Cheng and Nakagawa [6] and Ki et al. [12].

Remark 5. Euclidean space R^n which is defined by $x_1 = x_{n+2} + t$ is a totally umbilical space-like hypersurface of $S_1^{n+1}(c)$ in R_1^{n+2}, where $\{x_1, \ldots, x_{n+2}\}$ is the natural coordinate system in R_1^{n+2}. The mean curvature H satisfies $H^2 = c$.

Remark 6. We consider a family of space-like hypersurfaces $H^k(c_1) \times S^{n-k}(c_2)$ of $S_1^{n+1}(c)$ which is defined by

$$H^k(c_1) \times S^{n-k}(c_2) = \left\{(x, y) \in S_1^{n+1}(c) \subset R_1^{n+2} = R_1^{k+1} \times R^{n-k+1} : |x|^2 = -\frac{1}{c_1}, |y|^2 = \frac{1}{c_2}\right\},$$

where $c_1 < 0$, $c_2 > 0$ and $1/c_1 + 1/c_2 = 1/c$. When $k > 1$, it is not of non-negative curvature. The number of distinct principal curvatures of such a hypersurface is exactly two. A principal curvature is equal to $(c - c_1)^{1/2}$ with the multiplicity k and the other is equal to $(c - c_2)^{1/2}$ with multiplicity $n - k$. We can prove that

$$S < \frac{1}{2(n-1)}[n\{n^2H^2 - 2(n-1)c\} + (n-2)n|H|\{n^2H^2 - 4(n-1)c\}^{1/2}],$$

and $H^2 \geq c$ if we choose appropriate values c_1 and c_2. Hence, the assumption on sectional curvature in Theorem 2 is essential.

Remark 7. Hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ has non-negative curvature and satisfies $H^2 \geq c$ if we choose appropriate values c_1 and c_2. The squared norm S of the second fundamental form α satisfies $S = 1/2(n-1)[n\{n^2H^2 - 2(n-1)c\} + (n-2)n|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]$. Hence, the estimate of S in Theorem 2 is best possible.

2. Preliminaries

First of all, we review basic formulas on space-like hypersurfaces in a Lorentz space. Let (M', g') be an $(n+1)$-dimensional Lorentz space, i.e., an indefinite Riemannian manifold of index 1. Throughout this paper, manifolds are always assumed to be connected and geometric objects are assumed to be of class C^∞. For any point x in M' we choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_1, \ldots, e_n\}$ on a neighborhood of x. Here and in the sequel, the following convention on the range of indices will be used throughout this paper, unless otherwise stated:

$$A, B, \ldots = 0, 1, \ldots, n, \qquad i, j, \ldots = 1, \ldots, n.$$

Let $\{\omega_A\} = \{\omega_0, \omega_1, \ldots, \omega_n\}$ denote the dual frame fields of $\{e_A\}$ on M'. Then metric tensor g' of M' satisfies $g'(e_A, e_B) = \epsilon_A \delta_{AB}$, where $\epsilon_0 = -1$ and $\epsilon_j = 1$. The canonical forms ω_A and the connection forms ω_{AB} of M' satisfy the structure equations:

$$\mathrm{d}\omega_A + \sum \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$\mathrm{d}\omega_{AB} + \sum \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \tag{2.2}$$

$$\Omega'_{AB} = -\frac{1}{2} \sum \epsilon_C \epsilon_D R'_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

where $\Omega' = \Omega'_{AB}$ (resp. R'_{ABCD}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R') of M'. The components R'_{CD} of the Ricci tensor and the scalar curvature r' are given by

$$R'_{CD} = \sum_B \epsilon_B R'_{BCDB}, \tag{2.4}$$

$$r' = \sum_A \epsilon_A R'_{AA}, \tag{2.5}$$

respectively. The components $R'_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor R' are defined by

$$\sum_E \epsilon_E R'_{ABCD;E}\omega_E = \mathrm{d}R'_{ABCD} - \sum_E \epsilon_E (R'_{EBCD}\omega_{EA} + R'_{AECD}\omega_{EB} + R'_{ABED}\omega_{EC} + R'_{ABCE}\omega_{ED}). \quad (2.6)$$

A plane section P' of the tangent space T_xM' of M' at any point x is said to be *non-degenerate*, provided that $g_x|P'$ is non-degenerate. It is easily seen that P' is non-degenerate if and only if it has a basis $\{u, v\}$ such that $g(u, u)g(v, v) - g(u, v)^2 \neq 0$. The sectional curvature of the non-degenerate plane section P' spanned by u and v is denoted by $K'(P') = K'(u, v)$. The semi-definite Riemannian manifold M' is said to be of *constant curvature* if its sectional curvature $K'(P')$ is constant for all P' and for all points of M'. M' is called *a semi-definite space form* if it is of constant curvature. An m-dimensional semi-definite space form of constant curvature c and of index s is denoted by $M_s^m(c)$. The standard models of semi-definite space forms are the following three kinds: the semi-definite Euclidean space R_s^m, the semi-definite spherical space $S_s^m(c)$ or the semi-definite hyperbolic space $H_s^m(c)$, according to $c = 0$, >0 or <0. The Riemannian curvature tensor R_{ABCD} of the semi-definite space form $M_s^m(c)$ is given by

$$R_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \quad (2.7)$$

Now, let (M', g') be an $(n+1)$-dimensional Lorentz space and let M be an n-dimensional space-like hypersurface of M'. We choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_1 \dots, e_n\}$ in such a way that restricted to M, $e_1, \dots, e_n$ are tangent to M and the other is normal to M. Namely, $e_1, \dots, e_n$ are space-like vectors and the other e_0 is a time-like vector. Let $\{\omega_A\}$ be its dual frame field. Then the indefinite Riemannian metric tensor g' of M' is given by $g' = \sum_A \epsilon_A\omega_A \otimes \omega_A$. The connection forms on M' are denoted by ω_{AB}, where $\epsilon_0 = -1$ and $\epsilon_j = 1$.

Restricting these forms to the space-like hypersurface M in M', we have

$$\omega_0 = 0, \quad (2.8)$$

and the induced metric g of M is given by $g = \sum \omega_j \otimes \omega_j$. From (2.1) and (2.8) and the Cartan lemma, we have

$$\omega_{0i} = \sum_j h_{ij}\omega_j, h_{ij} = h_{ji}. \quad (2.9)$$

The quadratic form $\alpha = -\sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_0$ with values in the normal bundle and $H = 1/n\sum_{j=1}^n h_{jj}$ are called *second fundamental form* and *mean curvature* of the hypersurface M, respectively. When principal curvatures of M are constant, M is called *isoparametric*. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equation of M:

$$\mathrm{d}\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \quad (2.10)$$

$$\mathrm{d}\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \qquad \Omega_{ij} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \quad (2.11)$$

From (2.3) and (2.11), we have Gauss equation

$$R_{ijkl} = R'_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}). \tag{2.12}$$

Components R_{ij} of Ricci tensor and scalar curvature r of M are given by

$$R_{ij} = \sum_{k=1}^{n} R'_{kijk} - nHh_{ij} + \sum_{k=1}^{n} h_{ik}h_{kj}, \tag{2.13}$$

$$r = \sum_{j,k=1}^{n} R'_{kjjk} - n^2H^2 + S, \tag{2.14}$$

where $S = \sum_{i,j=1}^{n} h_{ij}^2$ denotes the squared norm of the second fundamental form of M.

By taking exterior differentiation of (2.9) and defining h_{ijk} by

$$\sum_k h_{ijk}\omega_k = \mathrm{d}h_{ij} - \sum_k (h_{kj}\omega_{ki} + h_{ik}\omega_{kj}), \tag{2.15}$$

we have Codazzi equation

$$h_{ijk} - h_{ikj} = R'_{0ijk}. \tag{2.16}$$

Similarly, defining h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = \mathrm{d}h_{ijk} - \sum_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk}), \tag{2.17}$$

and differentiating (2.15) exteriorly, we have

$$\begin{aligned}
&\sum_k \left\{ \mathrm{d}h_{ijk} \wedge \omega_k + h_{ijk}\left(-\sum_l \omega_{kl} \wedge \omega_l \right) \right\} \\
&\quad = -\sum_k \left[\mathrm{d}h_{kj} \wedge \omega_{ki} + h_{jk}\left\{ -\sum_l \omega_{kl} \wedge \omega_{li} - \frac{1}{2}\sum_{l,m} R_{kilm}\omega_l \wedge \omega_m \right\} \right. \\
&\qquad \left. + \mathrm{d}h_{ik} \wedge \omega_{kj} + h_{ik}\left\{ -\sum_l \omega_{kl} \wedge \omega_{lj} - \frac{1}{2}\sum_{l,m} R_{kjlm}\omega_l \wedge \omega_m \right\} \right].
\end{aligned}$$

Hence, we obtain Ricci formula for the second fundamental form of M:

$$h_{ijkl} - h_{ijlk} = -\sum_r (h_{ir}R_{rjkl} + h_{jr}R_{rikl}). \tag{2.18}$$

Now let us denote, by $R'_{ABCD;E}$, covariant derivative of R'_{ABCD}. Then, restricting on M, $R'_{0ijk;l}$ is given by

$$R'_{0ijk;l} = R'_{0ijkl} + R'_{0i0k}h_{jl} + R'_{0ij0}h_{kl} + \sum_m R'_{mijk}h_{ml}, \tag{2.19}$$

where R'_{0ijkl} denote the covariant derivative of R'_{0ijk} as a tensor on M so that

$$\sum_l R'_{0ijkl}\omega_l = \mathrm{d}R'_{0ijk} - \sum_l R'_{0ljk}\omega_{li} - \sum_l R'_{0ilk}\omega_{lj} - \sum_l R'_{0ijl}\omega_{lk}.$$

Next, we compute the Laplacian Δh_{ij} defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}. \tag{2.20}$$

From (2.16) and (2.18) it follows that

$$\begin{aligned}\Delta h_{ij} &= \sum_k h_{ikjk} + \sum_k R'_{0ijkk} = \sum_k h_{kijk} + \sum_k R'_{0ijkk}\\ &= \sum_k \left\{ h_{kikj} - \sum_l (h_{kl}R_{lijk} + h_{il}R_{lkjk}) + R'_{0ijkk} \right\}.\end{aligned}$$

From $h_{kikj} = h_{kkij} + R'_{0kikj}$, we obtain

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_k (R'_{0ijkk} + R'_{0kikj}) - \sum_{k,l} (h_{kl}R_{lijk} + h_{il}R_{lkjk}).$$

By (2.12) and (2.19) and the above equation, we obtain

$$\begin{aligned}\Delta h_{ij} &= \sum_k h_{kkij} + \sum_k (R'_{0ijk;k} + R'_{0kik;j}) - \sum_k \left(h_{jk}R'_{0i0k} + h_{kk}R'_{0ij0} + \sum_l h_{kl}R'_{lijk} \right)\\ &\quad - \sum_k \left(h_{ij}R'_{0k0k} + h_{kj}R'_{0ki0} + \sum_l h_{jl}R'_{lkik} \right) - \sum_{k,l} (R'_{lkjk} - h_{lk}h_{jk} + h_{kk}h_{jl})h_{il}\\ &\quad - \sum_{k,l} (R'_{lijk} - h_{kl}h_{ij} + h_{lj}h_{ik})h_{kl}\\ &= \sum_k h_{kkij} + \sum_k (R'_{0ijk;k} + R'_{0kik;j}) - \sum_k (h_{kk}R'_{0ij0} + h_{ij}R'_{0k0k})\\ &\quad - \sum_{k,l} (2h_{kl}R'_{lijk} + h_{jl}R'_{lkik} + h_{il}R'_{lkjk}) - nH\sum_l h_{il}h_{lj} + Sh_{ij}.\end{aligned} \tag{2.21}$$

The following Generalized Maximum Principle of Omori [16] and Yau [21] will play an important role in the proof of our Theorems (cf. [8,20]).

Generalized Maximum Principle. Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from above on M, then, for any $\epsilon > 0$, there exists a point $p \in M$ such that

$$|\nabla F(p)| < \epsilon, \qquad \Delta F(p) < \epsilon \quad \text{and} \quad \sup F - \epsilon < F(p). \tag{2.22}$$

3. Locally formulas

In this section, we assume that M' is an $(n+1)$-dimensional Lorentz space satisfying condition $(**)$ and M is a space-like hypersurface with constant mean curvature in M'. First of all, we calculate the Laplacian of the squared norm S of the second fundamental form α of M.

$$\frac{1}{2}\Delta S = \frac{1}{2}\Delta\left(\sum_{i,j} h_{ij}^2\right) = \sum_{i,j,k}(h_{ijk}h_{ij})_k = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ijkk}h_{ij}. \tag{3.1}$$

By (2.21), we have

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}\Bigg[\sum_k (R'_{0ij0;j} + R'_{0ijk;k}) - \sum_k (h_{kk}R'_{0ij0} + h_{ij}R'_{0k0k}) \\ & - \sum_{k,l}(2h_{kl}R'_{lijk} + h_{lj}R'_{lkik} + h_{li}R'_{kjkl}) - nH\sum_l h_{il}h_{lj} + Sh_{ij}\Bigg]h_{ij}.\end{aligned} \tag{3.2}$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij}(R'_{0kik;j} + R'_{0ijk;k}) - \left(\sum_{i,j} nHh_{ij}R'_{0ji0} + S\sum_k R'_{0k0k}\right) \\ & - \sum_{i,j,k,l} 2(h_{ij}h_{kl}R'_{lijk} + h_{li}h_{ij}R'_{lkjk}) - nHh_3 + S^2,\end{aligned} \tag{3.3}$$

where $h_3 = \sum_{j=1}^n \lambda_j^3$ and λ_j's are principal curvatures of M.

Next, we will choose $\{e_1, \ldots, e_n\}$ such that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{3.4}$$

By definition, we see

$$S = \sum_i \lambda_i^2.$$

Putting $\mu_j = \lambda_j - H$, we have

$$\sum_j \mu_j = 0, \qquad \sum_j \mu_j^2 = S - nH^2. \tag{3.5}$$

Therefore, for any j

$$(\lambda_j - H)^2 \le \frac{n-1}{n}(S - nH^2). \tag{3.6}$$

Since

$$\begin{aligned}
&\sum_{i,j,k}(R'_{0kik;j}+R'_{0ijk;k})h_{ij}\\
&\quad=\sum_{j,k}\lambda_j(R'_{0kjk;j}+R'_{0jjk;k})\\
&\quad\geq-\sum_k\left[|H|+\sqrt{\frac{n-1}{n}(S-nH^2)}\right]\left(\sqrt{\sum_j R'^2_{0kjk;j}}+\sqrt{\sum_j R'^2_{0jjk;k}}\right)\\
&\quad\geq-2\sqrt{n}\left[|H|+\sqrt{\frac{n-1}{n}(S-nH^2)}\right]|\nabla R'|,
\end{aligned}$$

we obtain

$$\sum_{i,j,k}(R'_{0kik;j}+R'_{0ijk;k})h_{ij}\geq-\frac{2}{\sqrt{n}}\left[|H|+\sqrt{\frac{n-1}{n}(S-nH^2)}\right]c_3. \tag{3.7}$$

By (3.4) and condition ($*$), we have

$$\begin{aligned}
&-\left(\sum_{i,j}nHh_{ij}R'_{0ij0}+S\sum_k R'_{0k0k}\right)\\
&\quad=-\sum_k nH\lambda_k\left(R'_{0kk0}-S\sum_k R'_{0kk0}\right)=\sum_k(S-nH\lambda_k)R'_{0kk0}=\sum_k(S-nH\lambda_k)\frac{c_1}{n}.
\end{aligned}$$

Hence it follows that

$$-\left(\sum_{i,j}nHh_{ij}R'_{0ij0}+S\sum_k R'_{0k0k}\right)=c_1(S-nH^2). \tag{3.8}$$

Since

$$\begin{aligned}
&-\sum_{i,j,k,l}(h_{ij}h_{kl}R'_{lijk}+h_{li}h_{ij}R'_{lkjk})\\
&\quad=-\sum_{j,k}(\lambda_j\lambda_k R'_{kjjk}-\lambda_k^2R'_{kjjk})=-\sum_{j,k}(\lambda_j\lambda_k-\lambda_k^2)R'_{kjjk}\\
&\quad=\frac{1}{2}\sum_{j,k}(\lambda_j-\lambda_k)^2R'_{kjjk}\geq\frac{c_2}{2}\sum_{j,k}(\lambda_j-\lambda_k)^2,
\end{aligned}$$

we obtain

$$-\sum_{i,j,k,l}(h_{ij}h_{kl}R'_{lijk}+h_{li}h_{ij}R'_{lkjk})\geq\frac{c_2}{2}\sum_{j,k}(\lambda_j-\lambda_k)^2=c_2(nS-n^2H^2). \tag{3.9}$$

Thus, substituting (3.7)–(3.9) into (3.3), we can prove the following lemma.

Lemma 1. *Let M' be an $(n+1)$-dimensional Lorentz space satisfying the condition $(**)$. If M is a space-like hypersurface with constant mean curvature H in M', then we have*

$$\frac{1}{2}\Delta S \geq -\frac{2}{\sqrt{n}}\left[|H| + \sqrt{\frac{n-1}{n}(S-nH^2)}\right]c_3 + (2nc_2 + c_1)(S-nH^2) + (S^2 - nHh_3). \tag{3.10}$$

In particular, if M' is locally symmetric, we have

$$\frac{1}{2}\Delta S \geq (2nc_2 + c_1)(S-nH^2) + (S^2 - nHh_3). \tag{3.11}$$

4. Proofs of Theorems

This section presents proofs of our theorems.

Proof of Theorem 1. Since M' is an $(n+1)$-dimensional locally symmetric Lorentz space satisfying the condition $(*)$, that is, for constants c_1 and c_2, we have

$$K'(u,v) = -\frac{c_1}{n} \tag{4.1}$$

for any space-like vector u and any time-like vector v and

$$K'(u_1,u_2) \geq c_2 \tag{4.2}$$

for any space-like vectors u_1 and u_2. Hence, we have, from (3.11)

$$\frac{1}{2}\Delta S \geq (2nc_2 + c_1)(S-nH^2) + (S^2 - nHh_3). \tag{4.3}$$

Let $B = \sum_i \mu_i^2$ and $B_3 = \sum_i \mu_i^3$. We have

$$B = S - nH^2, \quad B_3 = h_3 - 3HB - nH^3. \tag{4.4}$$

Hence, we have

$$\frac{1}{2}\Delta B \geq (2nc_2 + c_1)B + (B + nH^2)^2 - nH(B_3 + 3HB + nH^3), \tag{4.5}$$

because H is constant.

Let $a_1, \ldots, a_n$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = B$, then we can prove

$$\left|\sum_i a_i^3\right| \leq \frac{n-2}{\sqrt{n(n-1)}}B^{3/2}, \tag{4.6}$$

and the equality holds if and only if at least $n-1$ of the a_i's are equal.

Therefore, we infer

$$\frac{1}{2}\Delta B \geq B\left\{B - \frac{n-2}{\sqrt{n(n-1)}}n|H|B^{1/2} + nc - nH^2\right\}, \tag{4.7}$$

where $c = 2c_2 + (c_1/n)$. Since M' satisfies the condition $(*)$ and M has constant mean curvature, from (2.13), we know that the Ricci curvature of M is bounded from below. Since we do not know whether S is bounded yet, we consider a function F defined by, for any positive constant a, $F = -1/(\sqrt{B+a})$. We know that F is bounded because of $B \geq 0$. According to the Generalized Maximum Principle of Omori [16] and Yau [21] in Section 2, for any $\epsilon_m > 0$, there exists a point $p_m \in M$ such that

$$\Delta F(p_m) < \epsilon_m, \qquad |\nabla F|(p_m) < \epsilon_m, \qquad \sup F - \epsilon_m < F(p_m). \tag{4.8}$$

Since

$$\nabla_i F = \frac{1}{2}\frac{\nabla_i B}{(B+a)^{3/2}}, \qquad \Delta F = \frac{1}{2}\frac{\Delta B}{(B+a)^{3/2}} - \frac{3}{4}\frac{|\nabla B|^2}{(B+a)^{5/2}},$$

we have

$$|\nabla F| = \tfrac{1}{2}|F|^3|\nabla B|,$$

and

$$\tfrac{1}{2}|F|^4\Delta B = |F|\Delta F + 3|\nabla F|^2.$$

From (4.8), we infer

$$\tfrac{1}{2}|F(p_m)|^4\Delta B(p_m) < |F(p_m)|\epsilon_m + 3\epsilon_m^2.$$

For any positive constant $0 < \epsilon < 1$, letting $\epsilon_m \to 0$, we know that there exists a positive integer m_0 such that when $m > m_0$, $|F(p_m)|\epsilon_m + 3\epsilon_m^2 < \epsilon$ because F is a bounded function. According to (4.7) and the above inequalities, we infer

$$(1-\epsilon)B^2(p_m) - \frac{n-2}{\sqrt{n(n-1)}}n|H|B^{3/2}(p_m) + (nc - nH^2 - 2a\epsilon)B(p_m) - a^2\epsilon < 0.$$

Thus, we know that $\{B(p_m)\}$ is a bounded sequence. Since $\lim_{m\to\infty}F(p_m) = \sup F = -\inf(1/\sqrt{B+a}) = -(1/\sqrt{\sup B + a})$, we have $\lim_{m\to\infty}B(p_m) = \sup B$ from the definition of F. Hence, B is bounded. Since a and ϵ are any positive constants, we infer

$$\sup B\left\{\sup B - \frac{n-2}{\sqrt{n(n-1)}}n|H|\sup B^{1/2} + nc - nH^2\right\} \leq 0. \tag{4.9}$$

If $n^2H^2 < 4(n-1)c$ holds, then we have $c > 0$ and

$$\sup B - \frac{n-2}{\sqrt{n(n-1)}}n|H|\sup B^{1/2} + nc - nH^2 > 0.$$

Hence, we obtain $\sup B = 0$, that is, $B \equiv 0$. Thus, we infer that $S \equiv nH^2$ and M is totally umbilical.

If $n^2H^2 = 4(n-1)c$ holds, then we have $c \geq 0$ and

$$\sup B - \frac{n-2}{\sqrt{n(n-1)}}n|H|\sup B^{1/2} + nc - nH^2 = \left(\sup B^{1/2} - \frac{n-2}{2\sqrt{n(n-1)}}n|H|\right)^2 \geq 0.$$

Hence, from (4.9), we have $\sup B = 0$, that is, $B \equiv 0$ if $\sup B^{1/2} \neq (n-2)/(2\sqrt{n(n-1)})n|H|$. Thus, we have that either $B \equiv 0$, that is, $S \equiv nH^2$ and M is totally umbilical, or $\sup B^{1/2} = (n-2)/(2\sqrt{n(n-1)})n|H|$, namely, $\sup S = nc$. Thus, we complete the proof of (2) in Theorem 1.

If $n^2H^2 > 4(n-1)c$ and $c < 0$ hold, we know

$$\begin{aligned}
&\sup B - \frac{n-2}{\sqrt{n(n-1)}} n|H| \sup B^{1/2} + nc - nH^2 \\
&\quad = (\sup B^{1/2} - B_{\min}^{1/2})(\sup B^{1/2} - B_{\max}^{1/2}),
\end{aligned}$$

where $B_{\min}^{1/2} = ((n-2)n|H| - n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)}$ and $B_{\max}^{1/2} = ((n-2)n|H| + n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)}$. Since $B_{\min}^{1/2} = ((n-2)n|H| - n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)} < 0$ holds when $c < 0$, from (4.9), we infer either $\sup B = 0$, in this case, M is totally umbilical, or $0 < \sup B^{1/2} \leq ((n-2)n|H| + n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)}$. Hence, we know that the assertion (3) in Theorem 1 is true from $S = B + nH^2$.

If $n^2H^2 > 4(n-1)c$ and $c \geq 0$ hold, we also have

$$\begin{aligned}
&\sup B - \frac{n-2}{\sqrt{n(n-1)}} n|H| \sup B^{1/2} + nc - nH^2 \\
&\quad = (\sup B^{1/2} - B_{\min}^{1/2})(\sup B^{1/2} - B_{\max}^{1/2}).
\end{aligned}$$

(a) When $H^2 \geq c$, since $B_{\min}^{1/2} = ((n-2)n|H| - n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)} \leq 0$ holds, from (4.9), we infer either $\sup B = 0$, in this case, M is totally umbilical, or $0 < \sup B^{1/2} \leq ((n-2)n|H| + n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)}$.
(b) When $H^2 < c$, we have $B_{\min}^{1/2} = ((n-2)n|H| - n\sqrt{n^2H^2 - 4(n-1)c})/2\sqrt{n(n-1)} > 0$. Hence, we have, from (4.9), $\sup B = 0$, in this case, M is totally umbilical, or $B_{\min}^{1/2} \leq \sup B^{1/2} \leq B_{\max}^{1/2}$. Thus, we infer that the assertion (4) in Theorem 1 is true because of $S = B + nH^2$.

If $S \equiv n/2(n-1)[n^2H^2 - 2(n-1)c + (n-2)|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]$ holds, we know that these inequalities in the proof of Lemma 1 and (4.6) are equalities and $S > nH^2$. Hence, we have $n^2H^2 \geq 4(n-1)c$ from (1) in Theorem 1. Thus, we can infer that $n-1$ of the principal curvatures λ_i are equal. Since the mean curvature H is constant and S is constant, we infer that principal curvatures are constant on M. Thus, M is an isoparametric hypersurface with two distinct principal curvatures one of which is simple. This completes the proof of Theorem 1. □

Proof of Theorem 2. According to Gauss equation (2.12), we have

$$R_{jkkj} = R'_{jkkj} - (h_{jj}h_{kk} - h_{jk}h_{kj}) = R'_{jkkj} - h_{jj}h_{kk} = R'_{jkkj} - \lambda_j\lambda_k.$$

Hence, we obtain

$$R'_{jkkj} = R_{jkkj} + \lambda_j\lambda_k. \tag{4.10}$$

Therefore

$$\begin{aligned} &-2\sum_{i,j,k,l} h_{ij}h_{kl}(R'_{lijk} - h_{li}h_{ij}R'_{lkjk}) \\ &\quad = -2\sum_{j,k}(\lambda_j\lambda_k R'_{kjjk} - \lambda_k^2 R'_{kjjk}) = -2\sum_{j,k}(\lambda_j\lambda_k - \lambda_k^2)R'_{kjjk} = \sum_{j,k}(\lambda_j - \lambda_k)^2 R'_{kjjk} \\ &\quad = \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2 R'_{kjjk} + \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2(R_{kjjk} + \lambda_j\lambda_k) \\ &\quad \geq \frac{c_2}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2 + \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2(R_{kjjk} + \lambda_j\lambda_k). \end{aligned} \tag{4.11}$$

By making use of the same proof as in the proof of Lemma 1, we have

$$\frac{1}{2}\Delta S \geq (nc_2 + c_1)(S - nH^2) + (S^2 - nHh_3) + \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2(R_{kjjk} + \lambda_j\lambda_i). \tag{4.12}$$

Since the sectional curvature of M is not less than $-(c_2 + c_1/n)$, we have

$$\begin{aligned} &\frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2(R_{kjjk} + \lambda_j\lambda_i) \\ &\quad \geq -\frac{1}{2}\left(c_2 + \frac{c_1}{n}\right)\sum_{j,k}(\lambda_j - \lambda_k)^2 + \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2\lambda_j\lambda_k \\ &\quad = -\left(c_2 + \frac{c_1}{n}\right)(nS - n^2H^2) + (nHh_3 - S^2). \end{aligned} \tag{4.13}$$

Thus, we infer, from (4.12) and (4.13)

$$\frac{1}{2}\Delta S \geq (nc_2 + c_1)(S - nH^2) + (S^2 - nHh_3) + \frac{1}{2}\sum_{j,k}(\lambda_j - \lambda_k)^2(R_{kjjk} + \lambda_j\lambda_i) \geq 0. \tag{4.14}$$

From Theorem 1, we know that S is bounded. Applying the Generalized Maximum Principle to S, we have that there exists a point sequence $\{p_m\} \subset M$ such that

$$\limsup_{m\to\infty} \Delta S(p_m) \leq 0, \qquad \lim_{m\to\infty} |\nabla S|(p_m) = 0, \qquad \lim_{m\to\infty} S(p_m) = \sup S. \tag{4.15}$$

Hence, we have

$$\lim_{m\to\infty}\sum_{j,k}(\lambda_j - \lambda_k)^2\left\{R_{kjjk} + \left(c_2 + \frac{c_1}{n}\right)\right\}(p_m) = 0, \tag{4.16}$$

and

$$\lim_{m\to\infty}\sum_{j,k}(\lambda_j - \lambda_k)^2 R'_{kjjk} = c_2 \lim_{m\to\infty}\sum_{j,k}(\lambda_j - \lambda_k)^2. \tag{4.17}$$

Since the function $S = \sum_j \lambda_j^2$ is bounded, $\{\lambda_j(p_m)\}$ is a bounded sequence for any j. Thus, we can assume

$$\lim_{m\to\infty} \lambda_j(p_m) = \lambda_{j_0} \tag{4.18}$$

for any j, if necessary, we can take a subsequence of $\{\lambda_j(p_m)\}$. Hence, from (4.10) and (4.17), we have

$$\lim_{m\to\infty} \sum_{j,k} (\lambda_j - \lambda_k)^2 R_{kjjk}(p_m) = \lim_{m\to\infty} \sum_{j,k} (\lambda_j - \lambda_k)^2 (c_2 - \lambda_j\lambda_k)(p_m). \tag{4.19}$$

Therefore, we infer, from (4.16) and (4.19)

$$\lim_{m\to\infty} \sum_{j,k} (\lambda_j - \lambda_k)^2 (c - \lambda_j\lambda_i) = 0. \tag{4.20}$$

If $\lambda_{j_0} \neq \lambda_{k_0}$, from (4.17), we have

$$\begin{aligned}\lim_{m\to\infty} (c_2 - \lambda_j\lambda_k)(p_m) &= \lim_{m\to\infty} (R'_{kjjk} - \lambda_j\lambda_k)(p_m) \\ &= \lim_{m\to\infty} R_{kjjk}(p_m) \geq -\left(c_2 + \frac{c_1}{n}\right).\end{aligned}$$

Hence, in this case, we obtain

$$\lim_{m\to\infty} (c - \lambda_j\lambda_i)(p_m) \geq 0. \tag{4.21}$$

Thus, (4.20) and (4.21) yield, for any i and j

$$\lim_{m\to\infty} (-\lambda_i\lambda_j(p_m) + c)(\lambda_i - \lambda_j)^2(p_m) = 0. \tag{4.22}$$

By (4.18) and (4.22) we get

$$(-\lambda_{i_0}\lambda_{j_0} + c)(\lambda_{i_0} - \lambda_{j_0})^2 = 0 \tag{4.23}$$

for any i_0 and j_0. By the simple algebraic calculation, it is clear that at most two of $\{\lambda_{j_0}\}$'s are distinct. If all of $\{\lambda_{j_0}\}$'s coincides with each other, then, from (4.21), we have $c \geq \lambda_{j_0}^2 \geq 0$.

If two of the $\{\lambda_{j_0}\}$'s are distinct, which are denoted by λ and μ ($\lambda \neq \mu$). By (4.23), they satisfy

$$-\lambda\mu + c = 0. \tag{4.24}$$

Now let us denote by r and s the number of indices $\lambda_j(p_m) \to \lambda$ and $\lambda_j(p_m) \to \mu$, respectively. Then we want to assert that $r = 1$ or $s = 1$.

In fact, if $r \geq 2$ and $s \geq 2$ hold, it follows from $r \geq 2$ that there are distinct indices i and j such that $\lambda_i(p_m) \to \lambda$ and $\lambda_j(p_m) \to \lambda$ ($m \to \infty$) and hence we have

$$\lim_{m\to\infty} (-\lambda_i\lambda_j + c) = -\lambda^2 + c, \quad \text{for } i \neq j.$$

By (4.18) and (4.21), we obtain $c \geq \lambda^2 \geq 0$. Similarly, we have $c \geq \mu^2 \geq 0$, which implies that

$$c^2 \geq \lambda^2\mu^2.$$

On the other hand, since we see that $c = \lambda\mu$ in (4.24), it implies $\lambda^2 = \mu^2 = c$. Furthermore, it turns out to be $\lambda = \pm\mu$. Because they are distinct, it yields $c = \lambda\mu = -\lambda^2$. Hence we obtain $\lambda = \mu = 0$. This is impossible because of $\lambda \neq \mu$. Thus, our assertion is true.

Without loss of generality, we assume $r = 1$. Since $n > 2$, we have $s \geq 2$ and by the above discussion we obtain $0 < \mu^2 < c$. This finishes the proof of the first part of assertions in Theorem 2.

Next, we shall prove the second part of the assertions in Theorem 2. From the above assertion, together with (4.24) it follows that $\lambda^2 > c$. It is clear that we may assume that the mean curvature H is positive. Then λ and μ are positive because of $c > 0$ and (4.24). By defining c_3 and c_4 by $\lambda^2 = c - c_3$ and $\mu^2 = c - c_4$, respectively, we have $c_3 < 0$, $0 < c_4 < c$ and

$$(c - c_3)(c - c_4) = c^2, \quad \text{i.e.,} \quad \frac{1}{c_3} + \frac{1}{c_4} = \frac{1}{c}. \tag{4.25}$$

Since the mean curvature H is constant, we obtain

$$nH = \lambda + (n-1)\mu = (c - c_3)^{1/2} + (n-1)(c - c_4)^{1/2}, \quad \lambda\mu = c.$$

Hence, we have

$$\lambda^2 - nH\lambda + (n-1)c = 0.$$

Since $H^2 \geq c$ holds, we conclude

$$\lambda = \frac{c}{\mu} = \frac{nH + \sqrt{n^2H^2 - 4(n-1)c}}{2}, \quad \text{and} \quad \mu = \frac{nH - \sqrt{n^2H^2 - 4(n-1)c}}{2(n-1)}. \tag{4.26}$$

Hence, we infer

$$\begin{aligned} S &= \lambda^2 + (n-1)\mu^2 \\ &= \frac{1}{2(n-1)}[n\{n^2H^2 - 2(n-1)c\} + (n-2)n|H|\{n^2H^2 - 4(n-1)c\}^{1/2}]. \end{aligned} \tag{4.27}$$

It is a contradiction. Therefore

$$\lambda_{j_0} = \lambda$$

for any j. This implies

$$\sup S = nH^2.$$

From $S - nH^2 = B \geq 0$, we have $S \equiv nH^2$. Hence, M is totally umbilical. $\square$

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